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Order automorphisms on positive definite operators and a few applications ☆

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ABSTRACT

We first determine the order automorphisms of the set of all positive definite operators with respect to the usual order and to the so-called chaotic order. We then apply those results to the following problems: (1) description of all bijective transformations on the space of nonsingular density operators (quantum states) which preserve the Umegaki or the Belavkin–Staszewski relative entropy; (2) characterization of the logarithmic product as the essentially unique binary operation on the set of positive definite operators that makes it an ordered commutative group with respect to the chaotic order.

1. Introduction

In this paper, H denotes a complex Hilbert space with dimension $\dim H > 1$ and inner product $\langle ., . \rangle$. The space of all bounded linear operators acting on H is denoted by $B(H)$. Following the tradition of linear algebra rather than that of the theory of C^*-algebras, an element A of $B(H)$ is called positive

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semi-definite if $\langle Ax, x\rangle \geqslant 0$ holds for every $x \in H$. If A is positive semi-definite and invertible, we call it a positive definite operator. As usual, a self-adjoint operator is an element of $B(H)$ which equals its adjoint. The usual order between self-adjoint operators A, B on H is defined in the following way: we write $A \leqslant B$ if and only if $B - A$ is positive semi-definite, i.e., $\langle Ax, x\rangle \leqslant \langle Bx, x\rangle$ holds for every $x \in H$.

In the paper [7], we determined the order automorphisms of the cone $B(H)^+$ of all positive semi-definite operators on H as well as the order automorphisms of the space $B(H)_s$ of all self-adjoint operators on H (alternatively, see Section 2.5 in the book [9]). We proved that any bijective map ϕ on $B(H)^+$ which preserves the order in both directions (i.e., it has the property that for any $A, B \in B(H)^+$, the relation $A \leqslant B$ holds if and only if $\phi(A) \leqslant \phi(B)$) is necessarily of the form $\phi(A) = TAT^*$ ($A \in B(H)^+$) with some invertible bounded linear or conjugate-linear operator T on H. As for the order automorphisms of the space $B(H)_s$, we proved that they are all of the form $\phi(A) = TAT^* + X$ ($A \in B(H)_s$) where T is as above and $X \in B(H)_s$ is some fixed element. We emphasize that in those results 'a priori' we did not assume any sort of linearity of the considered automorphisms. We also mention that an interesting strengthening of the latter result can be found in [17].

Denote by $B(H)^+_{-1}$ the set of all positive definite operators on H. In the first part of the next section where we present our results we determine the order automorphisms of this set with respect to the usual order and also to the so-called chaotic order. After that we apply the results to the problems formulated in the abstract.

2. Results

We begin with the description of the structure of order automorphisms of $B(H)^+_{-1}$ with respect to the usual order $\leqslant$. We shall see that they are of the same form as the order automorphism of $B(H)^+$.

Theorem 1. *Let $\phi : B(H)^+_{-1} \to B(H)^+_{-1}$ be a bijective map with the property that*

$$A \leqslant B \Longleftrightarrow \phi(A) \leqslant \phi(B)$$

holds whenever $A, B \in B(H)^+_{-1}$. Then there exists an invertible bounded either linear or conjugate-linear operator $T : H \to H$ such that

$$\phi(A) = TAT^* \qquad (A \in B(H)^+_{-1}).$$

Proof. The simple idea of the proof is to reduce the problem to the case of the order automorphisms of $B(H)^+$. Pick any number $0 < \epsilon \leqslant 1$ and define

$$\psi(A) = \phi(A + \epsilon I) - \phi(\epsilon I)$$

for every $A \in B(H)^+$. One can readily verify that $\psi : B(H)^+ \to B(H)^+$ is a bijective map which preserves the order in both directions. It follows from the description of the order automorphisms of $B(H)^+$ that there exists an invertible bounded either linear or conjugate-linear operator T_ϵ on H such that $\phi(A + \epsilon I) - \phi(\epsilon I) = T_\epsilon A T_\epsilon^*$, or equivalently,

$$\phi(A + \epsilon I) = T_\epsilon A T_\epsilon^* + \phi(\epsilon I)$$

holds for all $A \in B(H)^+$. It follows that

$$\phi(B) = T_\epsilon B T_\epsilon^* - \epsilon T_\epsilon T_\epsilon^* + \phi(\epsilon I)$$

for all $B \geqslant \epsilon I$. Denoting $S_\epsilon = -\epsilon T_\epsilon T_\epsilon^* + \phi(\epsilon I)$, this can be rewritten as

$$\phi(B) = T_\epsilon B T_\epsilon^* + S_\epsilon \quad (B \geqslant \epsilon I). \tag{1}$$

We deduce that there is an invertible bounded either linear or conjugate-linear operator T_1 on H and a self-adjoint operators S_1 such that

$$\phi(B) = T_\epsilon B T_\epsilon^* + S_\epsilon \text{ and } \phi(B) = T_1 B T_1^* + S_1$$

both hold for $B \geqslant I$. This means that

$$T_\epsilon B T_\epsilon^* + S_\epsilon = T_1 B T_1^* + S_1 \tag{2}$$

holds for every $B \geqslant I$. Since any self-adjoint operator D on H is the difference of two operators B, C with $B, C \geqslant I$, we obtain from (2) that

$$T_\epsilon D T_\epsilon^* = T_1 D T_1^*$$

holds for every such operator D. In particular, we obtain that $S_\epsilon = S_1$. Going back to (1), we infer that

$$\phi(B) = T_1 B T_1^* + S_1 \quad (B \geqslant \epsilon I).$$

Since $0 < \epsilon \leqslant 1$ is arbitrary, it follows that $\phi(B) = T_1 B T_1^* + S_1$ holds for every $B \in B(H)_{-1}^+$. As ϕ maps onto $B(H)_{-1}^+$, it is easy to see that we necessarily have $S_1 = 0$ completing the proof of the theorem. □

The next result describes the chaotic order automorphisms of the set of all positive definite operators. The chaotic order denoted by $\ll$ is defined in the following way: for any $A, B \in B(H)_{-1}^+$ we write $A \ll B$ if $\log A \leqslant \log B$ (here log is the logarithmic function with the natural base). This relation has important applications relating to operator inequalities such as Löwner-Heinz- and Furuta-type inequalities (for some results see [4] and its references).

Theorem 2. *Let $\phi : B(H)_{-1}^+ \to B(H)_{-1}^+$ be a bijective map with the property that*

$$A \ll B \iff \phi(A) \ll \phi(B)$$

holds whenever $A, B \in B(H)_{-1}^+$. Then there exists an invertible bounded either linear or conjugate-linear operator $T : H \to H$ and a self-adjoint operator X such that

$$\phi(A) = e^{T(\log A)T^* + X} \qquad (A \in B(H)_{-1}^+).$$

Proof. Using continuous function calculus, consider the transformation

$$\psi(A) = \log \phi\left(e^A\right) \qquad (A \in B(H)_s).$$

It is easy to see that this map is an order automorphism of $B(H)_s$ with respect to the usual order $\leqslant$. Therefore, by the results in [7] mentioned in Section 1 we obtain that there is an invertible bounded linear or conjugate-linear operator T on H and a fixed self-adjoint operator X such that

$$\psi(A) = TAT^* + X \qquad (A \in B(H)_s).$$

Transforming ψ back to ϕ we obtain the statement. □

We now apply the above results to the problems mentioned in Section 1. From this point to Remark 6 we assume that $\dim H < \infty$.

Denote by $S(H)$ the space of all density operators on H. By definition, the elements of $S(H)$ are the positive semi-definite operators on H with unit trace. In the Hilbert space formalism of quantum mechanics, the elements of $S(H)$ represent the (mixed) states of a quantum system to which H is associated. One of the most fundamental concepts in quantum information theory is that of the von Neumann entropy [15,16]. Based on this notion, different concepts of quantum relative entropy were defined to measure distinguishability between quantum states. The most common such concept is due to Umegaki. For any pair $A, B \in S(H)$ of states, the Umegaki relative entropy $S(A||B)$ is defined by [1]

$$S(A||B) = \begin{cases} \operatorname{tr}[A(\log A - \log B)], & \text{if } \operatorname{supp} A \subset \operatorname{supp} B; \\ +\infty, & \text{otherwise.} \end{cases}$$

Here and in what follows tr stands for the usual trace functional and supp denotes the orthogonal complement of the kernel of a density operator. It is well-known that the quantity $S(A||B)$ is always nonnegative and equals zero if and only if $A = B$. In the paper [10], we have determined the structure of all bijective maps on $S(H)$ which preserve the Umegaki relative entropy. We proved that every such transformation ϕ is of the form $\phi(A) = UAU^*$ $(A \in S(H))$ with some unitary or antiunitary operator U on H. The result is related to the famous theorem of Wigner describing the form of the so-called quantum mechanical symmetry transformations (bijective maps on the space of all pure states which preserve the transition probability).

So, in [10] we have studied transformation on the whole space $S(H)$. However, in several problems in quantum information theory and quantum statistics where differential geometric considerations and corresponding strong analytical tools are applied, instead of the whole set of density operators only the set $\mathcal{M}(H)$ of invertible elements of $S(H)$ are considered. The reason is that from differential geometric point of view $\mathcal{M}(H)$ is a much more appropriate set, namely, a manifold (see, e.g., [6]). Therefore, the natural problem arises that what the corresponding relative entropy preserving transformations are. We note that our idea in [10] was to restrict the original transformation to the set of rank-one projections (pure states) and prove that it is a quantum mechanical symmetry transformation in the sense of Wigner. Clearly, this idea cannot be followed when considering only invertible density operators. However, we can apply Theorem 2 on the structure of the chaotic order automorphisms of $B(H)^+_{-1}$.

Theorem 3. *Let $\phi : \mathcal{M}(H) \to \mathcal{M}(H)$ be a bijective map on the set of all invertible density operators which preserves the Umegaki relative entropy, i.e., which satisfies*

$$S(\phi(A)||\phi(B)) = S(A||B)$$

for all $A, B \in \mathcal{M}(H)$. Then there is an either unitary or antiunitary operator U on H such that

$$\phi(A) = UAU^* \qquad (A \in \mathcal{M}(H)).$$

Before the proof we introduce the following notation. For any pair $x, y \in H$ of vectors, $x \otimes y$ denotes the operator (of rank at most one) defined by

$$(x \otimes y)z = \langle z, y\rangle x \qquad (z \in H).$$

Proof. Define a transformation $\psi : B(H)^+_{-1} \to B(H)^+_{-1}$ by the formula

$$\psi(A) = (\operatorname{tr} A)\phi\left(\frac{A}{\operatorname{tr} A}\right) \qquad \left(A \in B(H)^+_{-1}\right).$$

Clearly, ψ is an extension of ϕ from $\mathcal{M}(H)$ to $B(H)^+_{-1}$. Elementary computation shows that

$$\operatorname{tr}[\psi(A)(\log \psi(A) - \log \psi(B))] = \operatorname{tr}[A(\log A - \log B)] \tag{3}$$

holds for every $A, B \in B(H)^+_{-1}$.

[1] We note that in information theory log usually stands for the logarithm with base 2 while in operator theory, and hence in relation with the chaotic order, the logarithm usually has the natural base e. In this paper, we follow the latter tradition. Recall that changing the base of logarithm means only multiplication by a constant factor and therefore has no effect on the conclusions in Theorems 3 and 5.

We assert that $\psi : B(H)^+_{-1} \to B(H)^+_{-1}$ is bijective. To the injectivity let $A, B \in B(H)^+_{-1}$ be such that

$$(\operatorname{tr} A)\phi\left(\frac{A}{\operatorname{tr} A}\right) = (\operatorname{tr} B)\phi\left(\frac{B}{\operatorname{tr} B}\right).$$

As ϕ maps into $S(H)$, taking traces we obtain $\operatorname{tr} A = \operatorname{tr} B$. By the injectivity of ϕ we obtain $A/\operatorname{tr} A = B/\operatorname{tr} B$ and then $A = B$. To the surjectivity let $C \in B(H)^+_{-1}$ and define $A = (\operatorname{tr} C)\phi^{-1}(C/\operatorname{tr} C)$. It follows that $\operatorname{tr} A = \operatorname{tr} C$ and then we obtain $\phi(A/\operatorname{tr} A) = C/\operatorname{tr} C$ implying $(\operatorname{tr} A)\phi(A/\operatorname{tr} A) = C$.

Observe now that for any $B, B' \in B(H)^+_{-1}$ we have $B \ll B'$ if and only if

$$\operatorname{tr}[A(\log A - \log B')] \leqslant \operatorname{tr}[A(\log A - \log B)] \qquad \left(A \in B(H)^+_{-1}\right). \tag{4}$$

Indeed, this latter property is equivalent to

$$\operatorname{tr}[A \log B] \leqslant \operatorname{tr}[A \log B'] \qquad \left(A \in B(H)^+_{-1}\right). \tag{5}$$

Now, if $\log B \leqslant \log B'$, then for any $A \in B(H)^+_{-1}$ we have $A^{1/2}(\log B)A^{1/2} \leqslant A^{1/2}(\log B')A^{1/2}$ and taking trace we obtain (5). On the other hand, if we have (5), then taking limits under the trace we see that $\operatorname{tr}[A \log B] \leqslant \operatorname{tr}[A \log B']$ holds for every positive semi-definite operator A, too. Inserting $A = x \otimes x$ for an arbitrary vector $x \in H$ into that inequality we obtain $\langle \log Bx, x\rangle \leqslant \langle \log B'x, x\rangle$. This gives that $\log B \leqslant \log B'$.

Considering the characterization of $B \ll B'$ given in (4) and using the property (3) of ψ and its bijectivity, we deduce that

$$B \ll B' \iff \psi(B) \ll \psi(B')$$

holds for any $B, B' \in B(H)^+_{-1}$. It follows that ψ is a chaotic order automorphism of $B(H)^+_{-1}$. By Theorem 2 we obtain that ψ is of the form

$$\psi(A) = e^{T(\log A)T^* + X} \qquad \left(A \in B(H)^+_{-1}\right)$$

where T is an invertible bounded linear or conjugate-linear operator on H and X is a self-adjoint operator.

The Eq. (3) tells us that

$$\operatorname{tr}\left[e^{T(\log A)T^*+X}(T(\log A)T^* - T(\log B)T^*)\right] = \operatorname{tr}[A(\log A - \log B)]$$

holds for all $A, B \in B(H)^+_{-1}$. We compute

$$\begin{aligned}\operatorname{tr}\left[e^{T(\log A)T^*+X}(T(\log A)T^* - T(\log B)T^*)\right] &= \operatorname{tr}\left[e^{T(\log A)T^*+X}(T(\log A - \log B)T^*)\right]\\ &= \operatorname{tr}\left[T^* e^{T(\log A)T^*+X}T(\log A - \log B)\right]\end{aligned}$$

and hence obtain that

$$\operatorname{tr}[T^* e^{T(\log A)T^*+X}T(\log A - \log B)] = \operatorname{tr}[A(\log A - \log B)]$$

for all $A, B \in B(H)^+_{-1}$. Fixing A for a moment and letting B run we see that $\log A - \log B$ runs through the whole set of self-adjoint operators. Therefore, from the last displayed formula we obtain

$$T^* e^{T(\log A)T^*+X}T = A$$

for every $A \in B(H)^+_{-1}$. Inserting $A = I$ we have $T^*e^X T = I$, and inserting $A = eI$ we have $T^*e^{TT^*+X}T = eI$. Next we compute

$$T^*e^{TT^*+X}T = eI = eT^*e^XT = T^*e^{I+X}T$$

from which it follows that $TT^* + X = I + X$. Therefore, $TT^* = I$ implying that T is a unitary or antiunitary operator. Then, from $T^*e^XT = I$ we obtain $e^X = I$ showing that $X = 0$. Finally, denoting $U = T$ we deduce

$$\phi(A) = \psi(A) = e^{U(\log A)U^*} = Ue^{\log A}U^* = UAU^* \qquad (A \in \mathcal{M}(H))$$

and this completes the proof. $\square$

Beside the Umegaki relative entropy other important concepts of quantum relative entropy have also been introduced and studied. In what follows we consider the well-known Belavkin–Staszewski relative entropy S_{BS} (see [3] or [16]) and determine the structure of transformations which preserve it. For any $A, B \in S(H)$ the quantity $S_{BS}(A||B)$ is defined by

$$S_{BS}(A||B) = \begin{cases} \mathrm{tr}[A\log(A^{1/2}B^{-1}A^{1/2})], & \text{if } \mathrm{supp}\, A \subset \mathrm{supp}\, B; \\ +\infty, & \text{otherwise} \end{cases}$$

(here B^{-1} stands for the inverse of B on $\mathrm{supp}\, B$). In the description of transformations of $\mathcal{M}(H)$ preserving the Umegaki relative entropy we have applied the structure theorem of chaotic order automorphisms of $B(H)^+_{-1}$. Concerning maps preserving the Belavkin–Staszewski we use our result on the structure of the usual order automorphisms of $B(H)^+_{-1}$. First we present a lemma which is needed in the proof of the next theorem.

Lemma 4. *Let $B, B' \in B(H)^+_{-1}$. We have $B \leqslant B'$ if and only if*

$$\mathrm{tr}\left[A\log\left(A^{1/2}B'^{-1}A^{1/2}\right)\right] \leqslant \mathrm{tr}\left[A\log\left(A^{1/2}B^{-1}A^{1/2}\right)\right]$$

holds for every $A \in B(H)^+_{-1}$.

Proof. Suppose $B \leqslant B'$. Then we have $B'^{-1} \leqslant B^{-1}$ implying

$$A^{1/2}B'^{-1}A^{1/2} \leqslant A^{1/2}B^{-1}A^{1/2}.$$

As the logarithmic function is well-known to be operator monotone, we obtain that

$$\log\left(A^{1/2}B'^{-1}A^{1/2}\right) \leqslant \log\left(A^{1/2}B^{-1}A^{1/2}\right).$$

Multiplying by $A^{1/2}$ from the left and from the right and taking traces we easily obtain the "only if" part of the statement.

Suppose now that

$$\mathrm{tr}\left[A\log\left(A^{1/2}B'^{-1}A^{1/2}\right)\right] \leqslant \mathrm{tr}\left[A\log\left(A^{1/2}B^{-1}A^{1/2}\right)\right] \tag{6}$$

holds for every $A \in B(H)^+_{-1}$. From the proof of Theorem 3.5 in [5] we learn that for a fixed $B \in B(H)^+_{-1}$ the function $A \mapsto \mathrm{tr}\, A(\log A^{1/2}B^{-1}A^{1/2})$ is (norm-) continuous on the set of all positive semi-definite operators. Therefore, as the inequality (6) holds for every positive definite A, it follows that it holds for every positive semi-definite A, too. Inserting $A = x \otimes x$ for any unit vector $x \in H$ and computing the two sides of (6) we have

$$\log\langle B'^{-1}x, x\rangle \leqslant \log\langle B^{-1}x, x\rangle.$$

From this we easily deduce $B \leqslant B'$. $\square$

Using the above lemma we can prove the following theorem that shows that those bijective maps of $\mathcal{M}(H)$ which leave the Belavkin–Staszewski relative entropy invariant are again of the regular form, they are implemented by unitary–antiunitary operators on the underlying Hilbert space.

Theorem 5. *Let $\phi : \mathcal{M}(H) \to \mathcal{M}(H)$ be a bijective map which preserves the Belavkin–Staszewski relative entropy, i.e., which satisfies*

$$S_{BS}(\phi(A)||\phi(B)) = S_{BS}(A||B)$$

for every $A, B \in \mathcal{M}(H)$. Then there is an either unitary or antiunitary operator U on H such that

$$\phi(A) = UAU^* \qquad (A \in \mathcal{M}(H)).$$

Proof. Basically, we use the same general approach as in the proof of Theorem 3. Namely, we first define the transformation $\psi : B(H)^+_{-1} \to B(H)^+_{-1}$ by the formula

$$\psi(A) = (\operatorname{tr} A)\phi\left(\frac{A}{\operatorname{tr} A}\right) \qquad \left(A \in B(H)^+_{-1}\right).$$

Just as in the proof of the previous theorem it is easy to see that ψ extends ϕ from $\mathcal{M}(H)$ to $B(H)^+_{-1}$, it is bijective and has the property that

$$\operatorname{tr}\left[\psi(A)\log\left(\psi(A)^{1/2}\psi(B)^{-1}\psi(A)^{1/2}\right)\right] = \operatorname{tr}\left[A\log\left(A^{1/2}B^{-1}A^{1/2}\right)\right] \tag{7}$$

for all $A, B \in B(H)^+_{-1}$. Applying Lemma 4 we obtain that ψ is an order automorphism of $B(H)^+_{-1}$. Therefore, by Theorem 1, there exists an invertible bounded linear or conjugate-linear operator T on H such that

$$\psi(A) = TAT^* \qquad \left(A \in B(H)^+_{-1}\right).$$

Substituting this into (7) we obtain that

$$\operatorname{tr}\left[TAT^*\log\left((TAT^*)^{1/2}(TBT^*)^{-1}(TAT^*)^{1/2}\right)\right] = \operatorname{tr}\left[A\log\left(A^{1/2}B^{-1}A^{1/2}\right)\right]$$

for all $A, B \in B(H)^+_{-1}$. Using the continuity property mentioned in the proof of Lemma 4 we deduce that the same equality holds true whenever A is a rank-one projection and $B \in B(H)^+_{-1}$. Inserting $A = x \otimes x$ for any unit vector $x \in H$ and computing the two sides of the previous displayed formula we find

$$\|Tx\|^2 \log\langle B^{-1}x, x\rangle = \log\langle B^{-1}x, x\rangle.$$

Since this holds for every $B \in B(H)^+_{-1}$, we obtain that $\|Tx\|^2 = 1$ for every unit vector $x \in H$ showing that T is either a unitary or an antiunitary operator. The statement of the theorem now follows. □

Remark 6. As the structure of all bijective maps on the whole set $S(H)$ of density operators which preserve the Belavkin–Staszewski relative entropy is not known in the literature, below we describe it as a quite easy consequence of Theorem 5. We shall see that the general form of those maps is the same again, they are all implemented by unitary or antiunitary operators.

The sketch of the proof is as follows. We know that $S_{BS}(A||B) < \infty$ if and only if $\operatorname{supp} A \subset \operatorname{supp} B$. Therefore, ϕ preserves the inclusion of supports of the elements of $S(H)$ in both directions, i.e., we have $\operatorname{supp} A \subset \operatorname{supp} B$ if and only if $\operatorname{supp}\phi(A) \subset \operatorname{supp}\phi(B)$ for any $A, B \in S(H)$. This implies easily that ϕ preserves the rank of the elements of $S(H)$. In particular, ϕ preserves the nonsingular elements of $S(H)$ as well as the rank-one elements of $S(H)$. As a consequence, ϕ maps $\mathcal{M}(H)$ onto itself and preserves the rank-one projections. We can apply our previous theorem to obtain that the restriction of ϕ onto $\mathcal{M}(H)$ is implemented by a unitary or antiunitary operator U on H. Next, considering the transformation $\psi(.) = U^*\phi(.)U$ we obtain a bijective map on $S(H)$ which preserves the Belavkin–Staszewski relative entropy and has the additional property that it is the identity on $\mathcal{M}(H)$. Let P be a rank-one projection and set $Q = \psi(P)$ which is a rank-one projection, too. Pick unit vectors x, y from the range of P and Q respectively. For any element $B \in \mathcal{M}(H)$, computing the Belavkin–Staszewski relative entropy between P, B and between $\psi(P), \psi(B) = B$, we have $\log\langle B^{-1}x, x\rangle = \log\langle B^{-1}y, y\rangle$. Since this holds for every $B \in \mathcal{M}(H)$, we obtain that x and y are scalar multiples of each other. Consequently, we have $Q = P$. This means that ψ leaves invariant not only the elements of $\mathcal{M}(H)$ but also the rank-one projections. It then follows that ψ preserves also the supports of the elements of $S(H)$. If $A, B \in S(H)$ have the same support and $S_{BS}(P||A) = S_{BS}(P||B)$ holds for every rank-one projection P with $\operatorname{supp} P \subset \operatorname{supp} A = \operatorname{supp} B$, then we obtain $\log\langle A^{-1}x, x\rangle = \log\langle B^{-1}x, x\rangle$ for every unit vector $x \in \operatorname{supp} A = \operatorname{supp} B$. Clearly, this implies $A = B$. Applying this observation to an arbitrary $A \in S(H)$ and $B = \psi(A)$, since $S_{BS}(P||A) = S_{BS}(\psi(P)||\psi(A)) = S_{BS}(P||B)$ holds for any rank-one projection P, we obtain that $\psi(A) = A$. Consequently, it follows that $\phi(A) = UAU^*$ holds for every $A \in S(H)$ and this is what we have asserted.[2]

To conclude the applications of Theorems 1 and 2 relating to quantum relative entropies, let us emphasize that using methods similar to those we have applied in the proofs of Theorems 3 and 5 one can get further results concerning the structure of other transformations on $B(H)_{-1}^{+}$ which preserve certain other numerical quantities. For example, one can prove easily that any bijective map on $B(H)_{-1}^{+}$ which preserves Uhlmann fidelity is again implemented by a unitary or an antiunitary operator. See [8] for the corresponding result on maps on the whole set $S(H)$.

As the second area of applications of the results on order automorphisms of $B(H)_{-1}^{+}$, we next consider the operation called logarithmic product (from now on $\dim H$ can also be infinite). This is defined by the formula

$$A \odot B = e^{\log A + \log B} \qquad \left(A, B \in B(H)_{-1}^{+}\right).$$

The concept has originally emerged from computational geometry [1] but soon after serious applications have been found in the differential geometry of spaces of positive definite operators which is a large and active area of research in present days. We mention that the Log-Euclidean geometric mean on $B(H)_{-1}^{+}$ (defined for any finite collection $A_1, \dots, A_n \in B(H)_{-1}^{+}$ by $\exp(\frac{1}{n}\sum_{i=1}^{n} A_i)$) is viewed as the Fréchet mean (least square) corresponding to the logarithmic product. This mean has recently been studied in details with applications to medical imaging with DT-MRI [2,12,13].

It is clear that the logarithmic product $\odot$ makes $B(H)_{-1}^{+}$ a commutative group which is ordered under the chaotic order (i.e., for any $A, B \in B(H)_{-1}^{+}$ if $A \ll B$, then we have $A \odot C \ll B \odot C$). In the next theorem we show that this property essentially characterizes the logarithmic product. To the proof we need the following lemma.

Lemma 7. *Let R, S, T be invertible bounded linear or conjugate-linear operators on H such that*

$$RXR^* = SXS^* + TXT^*$$

holds for all self-adjoint operator X on H. Then S is a scalar multiple of T.

Proof. We use a rather easy and standard argument. First observe the following. Let $x, y, z \in H$ be nonzero vectors such that

$$x \otimes x + y \otimes y = z \otimes z$$

[2] We have been informed [14] that using a method we developed in [11] (completely different from the one we present here), G. Nagy managed to describe the form of all maps $\phi : S(H) \to S(H)$ preserving the Belavkin–Staszewski relative entropy even when the bijectivity of ϕ is not assumed.

Suppose x, y are linearly independent. Choosing a vector y_p which is orthogonal to y but not orthogonal to x, from the previous equality we have

$$\langle y_p, x\rangle x = \langle y_p, z\rangle z$$

showing that x is a scalar multiple of z. In a similar way we obtain that y is also a scalar multiple of z yielding that x is a scalar multiple of y.

Now pick any nonzero vector $x \in H$ and insert $X = x \otimes x$ into the equation $RXR^* = SXS^* + TXT^*$. We have

$$Rx \otimes Rx = Sx \otimes Sx + Tx \otimes Tx$$

and hence obtain that Sx is a scalar multiple of Tx for every $x \in H$ the scalar might depending on x. This means that for every $x \in H$ we have $Sx = \lambda_x Tx$ for some complex number λ_x. Picking linearly independent vectors $x, y \in H$, using the so-called local linear dependence of S and T as well as the additivity of the operators S, T we obtain

$$\lambda_{x+y}(Tx + Ty) = \lambda_{x+y}T(x + y) = S(x + y) = Sx + Sy = \lambda_x Tx + \lambda_y Ty.$$

As Tx, Ty are also linearly independent, we infer that $\lambda_x = \lambda_y$. For arbitrary two nonzero vectors $x, y \in H$, choosing a nonzero $z \in H$ such that x, z and y, z are both linearly independent we conclude $\lambda_x = \lambda_z = \lambda_y$. This shows that λ_x does not depend on x and hence we obtain that $S = \lambda T$ for some complex number λ. This proves the statement. $\square$

We now present the already mentioned characterization theorem of the logarithmic product.

Theorem 8. *Suppose that $\bullet$ is a binary operation on $B(H)^+_{-1}$ that makes it a commutative group which is ordered under the chaotic order. Then we have*

$$A \bullet B = e^{\log A + \log B - \log E} \qquad \left(A, B \in B(H)^+_{-1}\right)$$

where E is the unit of the group $(B(H)^+_{-1}, \bullet)$.

Proof. As $B(H)^+_{-1}$ is an ordered group with the operation $\bullet$ and the order $\ll$, it follows immediately that for any $B \in B(H)^+_{-1}$ the transformation $A \mapsto A \bullet B$ is a chaotic order automorphism of $B(H)^+_{-1}$. Therefore, by Theorem 2, for any $B \in B(H)^+_{-1}$ we have an invertible bounded linear or conjugate-linear operator $\varphi(B)$ on H and a self-adjoint operator $\psi(B)$ such that

$$A \bullet B = e^{\varphi(B)(\log A)\varphi(B)^* + \psi(B)} \qquad \left(A, B \in B(H)^+_{-1}\right).$$

Putting $A = E$, we have

$$e^{\log B} = B = E \bullet B = e^{\varphi(B)(\log E)\varphi(B)^* + \psi(B)}$$

which implies

$$\psi(B) = \log B - \varphi(B)(\log E)\varphi(B)^* \qquad \left(B \in B(H)^+_{-1}\right).$$

Hence we have

$$A \bullet B = e^{\varphi(B)(\log A - \log E)\varphi(B)^* + \log B} \qquad \left(A, B \in B(H)^+_{-1}\right). \tag{8}$$

Pick arbitrary self-adjoint operators R, S, T on H and compute $\log\left(e^R \bullet e^{\frac{S+T}{2}}\right) = \log\left(e^{\frac{S+T}{2}} \bullet e^R\right)$. Applying (8) we have

$$\begin{aligned}\log\left(e^{\frac{S+T}{2}}\bullet e^R\right)&=\varphi\left(e^R\right)\left(\frac{S+T}{2}-\log E\right)\varphi\left(e^R\right)^*+R\\&=\frac{1}{2}\left(\varphi\left(e^R\right)(S-\log E)\varphi\left(e^R\right)^*+R\right)+\frac{1}{2}(\varphi\left(e^R\right)(T-\log E)\varphi\left(e^R\right)^*+R)\\&=\frac{1}{2}\log\left(e^S\bullet e^R\right)+\frac{1}{2}\log\left(e^T\bullet e^R\right)\\&=\frac{1}{2}\log\left(e^R\bullet e^S\right)+\frac{1}{2}\log\left(e^R\bullet e^T\right)\\&=\frac{1}{2}\left(\varphi\left(e^S\right)(R-\log E)\varphi\left(e^S\right)^*+S\right)+\frac{1}{2}\left(\varphi\left(e^T\right)(R-\log E)\varphi\left(e^T\right)^*+T\right).\end{aligned}$$

On the other hand we have

$$\log\left(e^R\bullet e^{\frac{S+T}{2}}\right)=\varphi\left(e^{\frac{S+T}{2}}\right)(R-\log E)\varphi\left(e^{\frac{S+T}{2}}\right)^*+\frac{S+T}{2}.$$

Comparing the last two displayed formulae we deduce that

$$\frac{1}{2}\varphi(e^S)(R-\log E)\varphi(e^S)^*+\frac{1}{2}\varphi(e^T)(R-\log E)\varphi(e^T)^*=\varphi\left(e^{\frac{S+T}{2}}\right)(R-\log E)\varphi\left(e^{\frac{S+T}{2}}\right)^*$$

holds for arbitrary self-adjoint operators R, S, T on H. As $R-\log E$ runs through the set of all self-adjoint operators, by Lemma 7 we obtain that $\varphi(e^S)$ and $\varphi(e^T)$ are scalar multiples of each other. Since S, T have been arbitrary self-adjoint operators, it implies that for any two elements $B, C \in B(H)_{-1}^+$, the operators $\varphi(B)$ and $\varphi(C)$ are scalar multiples of each other. In particular, every $\varphi(B)$ is a scalar multiple of $\varphi(E)$. We show that $\varphi(E)$ is itself a scalar multiple of the identity. To see this, pick any $A \in B(H)_{-1}^+$ and compute

$$A=A\bullet E=e^{\varphi(E)(\log A-\log E)\varphi(E)^*+\log E}.$$

This implies

$$\log A-\log E=\varphi(E)(\log A-\log E)\varphi(E)^*.$$

Since $\log A-\log E$ runs through the set of all self-adjoint operators on H, we obtain that

$$X=\varphi(E)X\varphi(E)^*$$

holds for every self-adjoint operator X. This implies that $\varphi(E)$ is a scalar multiple of the identity. Indeed, this can easily be seen applying an argument similar to the last part of the proof of Lemma 7. Therefore, we have that for every $B \in B(H)_{-1}^+$, the operator $\varphi(B)$ is also a scalar multiple of the identity. Hence, we can rewrite (8) as

$$A\bullet B=e^{\lambda_B(\log A-\log E)+\log B}\qquad\left(A,B\in B(H)_{-1}^+\right)$$

with some positive number λ_B. By the commutativity of the operation $\bullet$, for any $A, B \in B(H)_{-1}^+$ we have

$$\lambda_B(\log A-\log E)+\log B=\lambda_A(\log B-\log E)+\log A.$$

Rearranging we obtain that

$$(\lambda_B-1)\log A-(\lambda_A-1)\log B=(\lambda_B-\lambda_A)\log E.$$

Suppose $\lambda_B \neq 1$ for some $B \in B(H)_{-1}^+$. The above displayed formula implies that for every $A \in B(H)_{-1}^+$, the self-adjoint operator $\log A$ belongs to the two-dimensional linear space spanned by $\log E$ and $\log B$. Since $\log A$ runs through the set of all self-adjoint operators, this is an obvious contradiction. Therefore, we have $\lambda_B = 1$ for every $B \in B(H)_{-1}^+$ and we finally infer that

$$A \bullet B = e^{\log A - \log E + \log B} \qquad \left(A, B \in B(H)^+_{-1}\right).$$

This completes the proof of the theorem. □

We emphasize that the set $B(H)^+_{-1}$ can be made a commutative group in many ways. Indeed, upon the pattern of the logarithmic product, consider any continuous bijection $\varphi :]0, \infty[\to]-\infty, \infty[$ and, using continuous function calculus, define the operation

$$A \star B = \varphi^{-1}(\varphi(A) + \varphi(B)) \qquad (A, B \in B(H)^+_{-1}).$$

Apparently, this makes $B(H)^+_{-1}$ a commutative group (the unit is $\varphi^{-1}(0)$).

After verifying Theorem 8 one may ask how many group structures the set $B(H)^+_{-1}$ may carry which are ordered by the usual order. The probably surprising but easy answer to the question is "none" (in the considered case $\dim H > 1$). This is the content of our last result.

Theorem 9. *There is no binary operation on $B(H)^+_{-1}$ which makes it an ordered commutative group under the usual order $\leqslant$.*

Proof. Suppose on the contrary that there is such an operation $\bullet$. We argue as in the proof of Theorem 8. For any $B \in B(H)^+_{-1}$, the transformation $A \mapsto A \bullet B$ is a bijective map on $B(H)^+_{-1}$ which is an order automorphism with respect to the usual order. By Theorem 1 we have an invertible bounded linear or conjugate-linear operator $\varphi(B)$ on H such that

$$A \bullet B = \varphi(B) A \varphi(B)^* \quad \left(A, B \in B(H)^+_{-1}\right).$$

We compute

$$\begin{aligned} &\varphi(B+C)A\varphi(B+C)^* = A \bullet (B+C) = (B+C) \bullet A \\ &\quad = \varphi(A)(B+C)\varphi(A)^* = \varphi(A)B\varphi(A)^* + \varphi(A)C\varphi(A)^* \\ &\quad = B \bullet A + C \bullet A = A \bullet B + A \bullet C = \varphi(B)A\varphi(B)^* + \varphi(C)A\varphi(C)^*. \end{aligned}$$

This means that we have

$$\varphi(B+C)A\varphi(B+C)^* = \varphi(B)A\varphi(B)^* + \varphi(C)A\varphi(C)^*$$

for any $A, B, C \in B(H)^+_{-1}$. Since every self-adjoint operator is the difference of two positive definite ones, we obtain that

$$\varphi(B+C)X\varphi(B+C)^* = \varphi(B)X\varphi(B)^* + \varphi(C)X\varphi(C)^*$$

holds for every self-adjoint X and all $B, C \in B(H)^+_{-1}$. Applying Lemma 7 we deduce that $\varphi(B)$ and $\varphi(C)$ are scalar multiplies of each other whenever $B, C \in B(H)^+_{-1}$. On the other hand, if E denotes the unit under the operation $\bullet$, we have

$$B = B \bullet E = \varphi(E)B\varphi(E)^*$$

for every $B \in B(H)^+_{-1}$. Just as above, we infer from this that $X = \varphi(E)X\varphi(E)^*$ holds for every self-adjoint operator X. We have mentioned in the proof of Theorem 8 that this implies that $\varphi(E)$ is a scalar multiple of the identity. Since $\varphi(B)$ is a scalar multiple of $\varphi(E)$, it now follows that for every $B \in B(H)^+_{-1}$, the operator $\varphi(B)$ is a scalar multiple of the identity. However, from $B = E \bullet B = \varphi(B)E\varphi(B)^*$ it would follow that every element of $B(H)^+_{-1}$ is a scalar multiple of the unit E which is an obvious contradiction. This completes the proof of the statement. □

Remark 10. We conclude the paper with an immediate corollary of Theorem 9 and a question.

Combining the observation given after the proof of Theorem 8 and the statement of Theorem 9 we obtain the following. There is no such continuous bijection $\varphi :]0, \infty[\to] - \infty, \infty[$ such that φ, φ^{-1} are both operator monotone.

It is clear that the group of all positive definite elements (i.e., invertible positive elements) in a commutative C^*-algebra forms an ordered commutative group with the usual product under the usual order. We believe it is an interesting problem to investigate the following question. If the set of all positive definite elements in a C^*-algebra $\mathcal{A}$ can be made an ordered commutative group with some operation under the usual order, then the full algebra $\mathcal{A}$ is necessarily commutative?

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